

Factor ordering problem in stochastic inflation

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The stochastic approach to inflation suffers from ambiguities due to the arbitrary choice of the time variable and due to the choice of the factor ordering in the corresponding Fokker-Planck equation. Here it is shown that both ambiguities can be removed if we require that the factor ordering should be set in such a way that physical results are invariant with respect to time reparametrizations. This requirement uniquely selects the so-called Ito factor ordering. Additional ambiguities associated with non-trivial kinetic terms of the scalar fields are also discussed, as well as ways of constraining these ambiguities. [S0556-2821(99)08212-0]

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I. INTRODUCTION

Quantum fluctuations of the inflaton field ϕ play an important role in the inflationary scenarios. These fluctuations can be pictured as a random walk of ϕ superimposed on the deterministic classical evolution and can be described in terms of a probability distribution satisfying the Fokker-Planck (FP) equation [1–7]. This approach, however, suffers from significant ambiguities: the resulting probabilities depend on the arbitrary choice of the time variable t and on the ordering of the non-commuting factors in the FP equation [6–8].

The origin of the dependence on the choice of t has been extensively discussed in the literature [7–10]. In essence, the problem is that physical volumes of regions with the field ϕ in any given range grow unboundedly with time. The relative probabilities of different values of ϕ are given by the corresponding volume ratios, and we have the usual ambiguities arising when one tries to compare infinities. I suggested a possible resolution of this problem in Ref. [11] and will have nothing more to say about it here.

In this paper we shall concentrate on probability distributions for which the divergence of the physical volume is not relevant. These include the coordinate (rather than physical) volume distribution for ϕ and the physical volume distribution in models where inflation is not eternal. There are no infinities to deal with in these cases, and one might expect that there will also be no ambiguities. One finds, however, that there is still some dependence of the results on the choice of t , as well as on the factor ordering.

It has been argued in Ref. [8] that the factor ordering ambiguity represents uncertainties inherent in the stochastic approach. The resulting uncertainties in the probabilities have been estimated as $O(H^2)$, where H is the inflationary expansion rate and I use Planck units throughout the paper. In models of new inflation with $H \ll 1$ the uncertainties are small. Moreover, the uncertainties due to the choice of t are also $O(H^2)$, and it has been argued in [8] that such uncertainties are acceptable since they do not go beyond the factor ordering ambiguity (which is presumably associated with the limitations of the FP equation itself).

In the present paper I am going to take an alternative approach and require that the factor ordering should be set in

such a way that appropriately chosen probability distributions are invariant with respect to time reparametrization. I am going to show that such an ordering does indeed exist (the so-called Ito ordering) and suggest that it is the correct factor ordering to be used in the FP equation. This argument is presented in Sec. III, after reviewing the FP equation in Sec. II.

In Sec. IV, I consider models with non-trivial kinetic terms for ϕ which give rise to an additional factor-ordering ambiguity. I am going to argue that much of this ambiguity can be removed by imposing some reasonable requirements on the form of the FP equation. The conclusions of the paper are briefly summarized in Sec. V.

II. FOKKER-PLANCK EQUATION

We shall consider a model with several inflaton fields ϕ^a , $a = 1, \dots, N$, described by the Lagrangian

$$L = \frac{1}{2} \partial_\mu \phi^a \partial^\mu \phi^a - V(\phi). \quad (1)$$

The evolution of ϕ^a during inflation is described by the probability distribution $P(\phi, t) d^N \phi$ which is interpreted, up to a normalization, as the coordinate comoving volume of regions with specified values of ϕ^a in the intervals $d\phi^a$ at time t . The FP equation for $P(\phi, t)$ has the form

$$\frac{\partial P}{\partial t} = - \frac{\partial J^a}{\partial \phi^a}, \quad (2)$$

where the flux $J_a(\phi, t)$ is given by

$$J_a = -D(\phi)^{1/2+\beta} \frac{\partial}{\partial \phi^a} [D(\phi)^{1/2-\beta} P] + v_a(\phi) P. \quad (3)$$

Here,

$$D(\phi) = H(\phi)^{\alpha+2} / 8\pi^2 \quad (4)$$

is the diffusion coefficient,

$$H(\phi) = [8\pi V(\phi)/3]^{1/2} \quad (5)$$

is the inflationary expansion rate, and

$$v_a(\phi) = -\frac{1}{4\pi} H(\phi)^{\alpha-1} \frac{\partial}{\partial \phi^a} H(\phi) \quad (6)$$

is the “drift” velocity of the slow roll. ϕ -space indices are raised and lowered using the flat metric δ_{ab} ; hence, $J^a = J_a$, etc.

The parameter β in Eq. (3) represents the ambiguity in the ordering of the non-commuting factors $D(\phi)$ and $\partial/\partial\phi^a$. $\beta=0$ and $\beta=1/2$ correspond to the so-called Stratonovich and Ito factor orderings, respectively. The parameter α in Eqs. (4),(6) represents the freedom of choosing the time variable t which is assumed to be related to the proper time of comoving observers τ by

$$dt = [H(\phi(\tau))]^{1-\alpha} d\tau, \quad (7)$$

where the functions $\phi(\tau)$ are taken along the observer’s world lines. Hence, $\alpha=1$ corresponds to the proper time parametrization $t=\tau$ and $\alpha=0$ corresponds to using the logarithm of the scale factor as a time variable.

The FP equation applies in the region of ϕ -space enclosed by the thermalization boundary S_* where the conditions of slow roll are violated,

$$\left| \frac{\partial H}{\partial \phi^a}(\phi_*) \right| \sim 2\pi H(\phi_*). \quad (8)$$

Here, $\phi_* \in S_*$. The boundary condition on P requires that diffusion vanish on S_* :

$$n^a \frac{\partial}{\partial \phi^a} [D(\phi)^{1/2-\beta} P] = 0, \quad (9)$$

where n^a is the normal to S_* [12]. Since Eq. (8) is an order-of-magnitude relation, the exact location of the surface S_* depends on the choice of a constant of order 1. Although this introduces an ambiguity in the calculation of P , we note that diffusion is small in the region dominated by the slow roll, and the ambiguity in the choice of S_* (which necessarily lies in that region) does not significantly influence the solution of the FP equation [13].

I have assumed for simplicity that we are dealing with inflation of the “new” type. In the case of “chaotic” inflation we would have another boundary where $V(\phi) \approx 1$. Quantum gravity effects become important at this Planck boundary, and it is not clear what kind of boundary condition has to be imposed there.

The FP equation for the physical volume distribution $\tilde{P}(\phi, t)$, defined as the physical volume of regions with ϕ^a in the intervals $d\phi^a$ at time t , has the form

$$\frac{\partial \tilde{P}}{\partial t} = -\frac{\partial \tilde{J}^a}{\partial \phi^a} + 3H^\alpha \tilde{P}, \quad (10)$$

where

$$\tilde{J}_a = -D(\phi)^{1/2+\beta} \frac{\partial}{\partial \phi^a} [D(\phi)^{1/2-\beta} \tilde{P}] + v_a(\phi) \tilde{P} \quad (11)$$

and the last term in Eq. (10) represents the growth of the physical volume due to the expansion of the universe. The factor-ordering parameter β could in principle be different for the coordinate and physical volume distributions.

III. ITO FACTOR ORDERING AND THE TIME REPARAMETRIZATION INVARIANCE

Let us now identify probability distributions that can be expected to be invariant under time reparametrization. The functions $P(\phi, t)$, $\tilde{P}(\phi, t)$ are not suitable for this role, since they give distributions for ϕ on surfaces of constant t . Different choices of the time variable t result in different surfaces and different distributions. Mathematically, this is evident from the fact that the corresponding FP equations explicitly depend on the parameter α .

Let us consider a large comoving region which is defined by a spacelike hypersurface Σ at some initial time $t=0$. We shall consider a family of time variables parametrized by α as in Eq. (7) assuming however that the surfaces $t=0$ coincide with Σ for all these variables. Different parts of our comoving region will have different evolution histories and will thermalize with different values of $\phi \in S_*$. At any time t there will be parts of the region that are still inflating, but in the limit $t \rightarrow \infty$ all the comoving volume will be thermalized, except a part of measure zero.

Let us introduce the distribution $p(\phi_*) dS_*$ which maps the ϕ -space thermalization boundary S_* onto our comoving volume. It is defined as the fraction of the comoving volume that is going to thermalize at $\phi_* \in S_*$ in the surface element dS_* (at any time). The distribution $p(\phi_*)$ is defined without reference to any particular time variable, and we can expect it to be invariant with respect to time reparametrization. Using Eq. (3) for the flux and the boundary condition (9) we can express $p(\phi_*)$ as

$$p(\phi_*) = n^a(\phi_*) v_a(\phi_*) \int_0^\infty P(\phi_*, t) dt, \quad (12)$$

where $v_a(\phi)$ is given by Eq. (6). Now I am going to show that with Ito factor ordering, $\beta = -1/2$, $p(\phi_*)$ is indeed independent of the parameter α .

Introducing

$$\psi(\phi) = H^{\alpha-1}(\phi) \int_0^\infty P(\phi, t) dt \quad (13)$$

and integrating the FP equation (2) and the boundary condition (9) over t , we obtain

$$\frac{\partial}{\partial \phi^a} \left[\frac{1}{8\pi^2} \frac{\partial}{\partial \phi^a} (H^3 \psi) - \frac{1}{4\pi} \frac{\partial H}{\partial \phi^a} \psi \right] = -P_0(\phi), \quad (14)$$

$$n^a \frac{\partial}{\partial \phi^a} (H^3 \psi) = 0 \quad (\phi \in S_*). \quad (15)$$

Here, $P_0(\phi) = P(\phi, 0)$ is the initial distribution at $t=0$ and I have used $\beta = -1/2$ in Eq. (14). The thermalization boundary distribution $p(\phi_*)$ can also be expressed in terms of ψ ,

$$p(\phi_*) = -\frac{1}{4\pi} \frac{\partial H}{\partial \phi^a} n^a \psi(\phi_*). \quad (16)$$

The function $\psi(\phi)$ is uniquely determined by Eq. (14) with the boundary condition (15). It can then be used in Eq. (16) to evaluate the distribution $p(\phi_*)$. Note that the parameter α has been absorbed in the definition (13) of $\psi(\phi)$ and does not appear in Eqs. (14)–(16). This shows that $p(\phi_*)$ is independent of time parametrization.

It is not difficult to verify that the parameter α drops out of the equations for $p(\phi_*)$ only in the case of $\beta = -1/2$ corresponding to Ito factor ordering. We conclude, therefore, that Ito ordering is the correct choice to be used in the FP equation [14].

Turning now to the physical volume FP equation (10), we consider models in which inflation is not eternal; that is, all eigenvalues of the operator on the right hand side of Eq. (10) are negative. Then, at $t \rightarrow \infty$ all physical volume should be thermalized, except perhaps a part of measure zero. We can define the physical volume distribution $\tilde{p}(\phi_*) dS_*$, up to a normalization, as the physical volume of the regions which thermalized with a given $\phi_* \in S_*$ in the surface element dS_* . The physical volume is measured at the time of thermalization; so $\tilde{p}(\phi_*)$ is a mapping of S_* onto the thermalization hypersurface Σ_* which separates the inflating and thermalized regions of spacetime. When inflation is not eternal, the distribution $\tilde{p}(\phi_*)$ should be well defined and independent of the time parametrization. Now, it is easily seen that Eqs. (12)–(16) for the coordinate volume distribution can be applied to $\tilde{p}(\phi_*)$ as well, with only a trivial modification due to the last term in Eq. (10). The same argument goes through, and we conclude again that $\tilde{p}(\phi_*)$ is invariant under time reparametrizations only with the choice of Ito factor ordering, $\beta = -1/2$. This argument cannot be directly applied to models of eternal inflation, but it is natural to expect that the same factor ordering will apply in such models as well.

IV. DIFFUSION ON A ϕ -SPACE WITH A NON-TRIVIAL METRIC

Let us now consider models with non-trivial kinetic terms in the scalar field Lagrangian,

$$L = \frac{1}{2} K_{ab}(\phi) \partial_\mu \phi^a \partial^\mu \phi^b - V(\phi). \quad (17)$$

Such models arise, in particular, in the context of “modular inflation” [15]. The matrix $K_{ab}(\phi)$ has the meaning of the metric on the space of ϕ^a (the moduli space). We shall assume that the expansion rate $H(\phi)$ is small compared to the characteristic scale $\Delta\phi$ on which $K_{ab}(\phi)$ vary in ϕ -space. In

modular inflation, we expect $\Delta\phi \sim 1$; so this condition is satisfied when the scale of inflation is well below the Planck scale, $H(\phi) \ll 1$.

The fields ϕ^a can be thought of as coordinates in ϕ -space, and the form of the FP equation should be invariant with respect to the transformations

$$\phi^a \rightarrow \phi^{a'}(\phi^b). \quad (18)$$

Coordinates can always be chosen so that $K_{ab} = \delta_{ab}$ at any point in ϕ -space. We then expect that in the vicinity of that point the equation will take the form (2),(3), possibly with corrections suppressed by some powers of $H/\Delta\phi$. These conditions are satisfied by the simplest covariant generalization of the flat-metric equations (2),(3),

$$\begin{aligned} \frac{\partial P}{\partial t} &= \frac{1}{\sqrt{K}} \frac{\partial}{\partial \phi^a} \left\{ \sqrt{K} K^{ab} \left[\frac{\partial}{\partial \phi^b} (DP) - v_b P \right] \right\} \\ &= \nabla_a [\nabla^a (DP) - v^a P] = \nabla_a J^a. \end{aligned} \quad (19)$$

Here, $K^{ab}(\phi)$ is the contravariant metric satisfying

$$K^{ab}(\phi) K_{bc}(\phi) = \delta_c^a, \quad (20)$$

$K = \det(K_{ab})$, ∇_a is a covariant derivative with respect to ϕ^a , and I have used Ito factor ordering $\beta = -1/2$ in Eq. (3). Note that the conservation of probability,

$$\frac{\partial}{\partial t} \int P \sqrt{K} d^N \phi = - \int_{S_*} J^a dS_a, \quad (21)$$

follows immediately from Eq. (19).

We note, however, that the conditions of covariance and correspondence with the flat-metric form alone do not fix the form of the equation uniquely. One could, for example, replace the covariant derivative in Eq. (19) by a more general expression

$$\nabla^a \rightarrow \nabla_a + \xi H^2 \mathcal{R} \nabla^a + \xi' H^2 \mathcal{R}^{ab} \nabla_b + \dots, \quad (22)$$

where $\mathcal{R}(\phi)$ and $\mathcal{R}^{ab}(\phi)$ are respectively the scalar curvature and the Ricci tensor of the ϕ -space and ξ, ξ' are numerical coefficients. The powers of H in the additional terms in Eq. (22) are determined by dimensionality. Since $\mathcal{R} \sim (\Delta\phi)^{-2}$, these terms are suppressed by a factor $(H/\Delta\phi)^2$. Hence, Eq. (19) can be used as an approximate FP equation in models with $H \ll \Delta\phi$.

V. SUMMARY AND DISCUSSION

The most satisfactory way to determine the factor ordering in the FP equation would be to derive it from first principles. At present we do not have such a derivation. The approach I took in this paper is phenomenological: the factor ordering is determined by imposing some physically reasonable requirements on the form of the FP equation. I have shown that for a scalar field model with a flat metric in ϕ -space, $K_{ab}(\phi) = \delta_{ab}$, the factor ordering is uniquely de-

terminated by requiring that physical results should not depend on the arbitrary choice of the time variable. This approach selects the Ito factor ordering.

Additional ambiguities arising in models with a non-trivial metric $K_{ab}(\phi)$ are constrained by requiring (i) covariance with respect to coordinate transformations in ϕ -space and (ii) correspondence with the flat metric case. This fixes the form of the FP equation up to factors of the order of

$(H/\Delta\phi)^2$ in the diffusion term, where $\Delta\phi$ is the characteristic scale of variation of $K_{ab}(\phi)$.

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